HODGE STRUCTURES ASSOCIATED TO $SU(p,1)$

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Abstract. Let A be a general member of a PEL family of abelian varieties such that the semisimple part of the Hodge group of A is a product of copies of $SU(p,1)$ for some $p > 1$. We show that any effective Tate twist of a Hodge structure occurring in the cohomology of a power of A is isomorphic to a Hodge structure in the cohomology of some abelian variety.

1. Introduction

We say that a (rational) Hodge structure of weight n,

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

is *effective* if $V^{p,q} = 0$ unless p, $q \geq 0$, and we say that it is *motivic* (or *geometric*) if it is isomorphic to a Hodge substructure of $H^n(X, \mathbb{Q})$ for some smooth, projective variety X over $\mathbb{C}$. For $m \in \mathbb{Z}$, the *Tate twist* $V(m)$ is the Hodge structure of weight $n - 2m$ given by $V(m)^{p,q} = V^{p+m,q+m}$.

The general Hodge conjecture as formulated by Grothendieck [11] implies that any effective Tate twist of a motivic Hodge structure is again motivic. In a series of papers [2–8] we have shown for a large class of abelian varieties that every effective Tate twist of a Hodge structure contained in the cohomology of one of these abelian varieties is isomorphic to a Hodge structure occurring in the cohomology of some abelian variety. We have used this to prove the general Hodge conjecture in some cases. We have also shown the existence of a Hodge structure M which occurs in the cohomology of an abelian variety, such that $M(1)$ is effective but does not occur in the cohomology of *any* abelian variety [5, Theorem 5.5, p. 926].

Our earlier results apply to abelian varieties of type IV in only a few cases—namely, when the Hodge group is semisimple [2], or when the abelian variety is of CM-type [7], or, when the semisimple part of the Hodge group is a product of groups of the form $SU(p+1,p)$ [8]. In this paper we extend these results to any power of a general member A of a PEL-family of abelian varieties of type IV, such that the semisimple part of the Hodge group of A is a product of copies of $SU(p,1)$ for some $p > 1$; see Theorem 11 for the precise statement.

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We draw attention to two new features of this paper. First, the concept of *semidomination* (Definition 4) generalizes the concept of *weak self-domination* introduced in [8], and allows us to work with the semisimple part of the Hodge group of an abelian variety whose Hodge group is neither semisimple nor commutative (see Theorem 8). Second, families of abelian varieties which are *not* of PEL-type play a critical role, even though our main concern is a general member of a PEL-family.

Notations and conventions. All representations are finite-dimensional and algebraic. The derived group of a group G is denoted by G'. All abelian varieties are over $\mathbb{C}$. For an abelian variety A, we let

$$D(A) := \operatorname{End}(A) \otimes \mathbb{Q}$$

be its endomorphism algebra, $L(A)$ its Lefschetz group, $G(A)$ its Hodge group, $L'(A)$ the derived group of $L(A)$, and, $G'(A)$ the derived group of $G(A)$. For a finite field extension E of a field F, we let $\operatorname{Res}_{E/F}$ be the restriction of scalars functor, from varieties over E to varieties over F. The center of a group G is denoted by $Z(G)$. For a topological group G, we denote by G^0 the connected component of the identity.

2. Preliminaries

2.1. **Kuga fiber varieties.** We briefly recall Kuga's construction of families of abelian varieties [12]; full details may be found in Satake's book [20].

Let G be a connected, semisimple, linear algebraic group over $\mathbb{Q}$ with identity 1. Assume that G is of hermitian type, and has no nontrivial, connected, normal $\mathbb{Q}$-subgroup H with $H(\mathbb{R})$ compact. Then $X := G(\mathbb{R})^0/K$ is a bounded symmetric domain for a maximal compact subgroup K of $G(\mathbb{R})^0$. Let $\mathfrak{g} := \operatorname{Lie} G$ be the Lie algebra of G, $\mathfrak{k} := \operatorname{Lie} K$, and let $\mathfrak{g}_\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. Differentiating the natural map $\nu\colon G(\mathbb{R})^0 \to X$ induces an isomorphism of $\mathfrak{p}$ with $T_o(X)$, the tangent space of X at $o = \nu(1)$, and there exists a unique $H_0 \in Z(\mathfrak{k})$, called the *H-element* at o, such that $\operatorname{ad} H_0|\mathfrak{p}$ is the complex structure on $T_o(X)$.

Let β be a nondegenerate alternating form on a finite-dimensional rational vector space V. The symplectic group $Sp(V, \beta)$ is a $\mathbb{Q}$-algebraic group of hermitian type; the associated symmetric domain is the *Siegel space*

$$\begin{aligned}\mathfrak{S}(V,\beta) := \{\, J \in GL(V_\mathbb{R}) \mid J^2 = -I \text{ and} \\ \beta(x, Jy) \text{ is symmetric, positive definite}\,\}.\end{aligned}$$

$Sp(V, \beta)$ acts on $\mathfrak{S}(V,\beta)$ by conjugation. The H-element at $J \in \mathfrak{S}(V,\beta)$ is $J/2$.

Let $\rho\colon G \to Sp(V, \beta)$ be a representation defined over $\mathbb{Q}$. We say that ρ satisfies the *H_1-condition* relative to the H-elements H_0 and $H_0' = J/2$ if

$$[d\rho(H_0) - H_0', d\rho(g)] = 0 \qquad \text{for all } g \in \mathfrak{g}_\mathbb{R}. \tag{2.1}$$

The stronger condition

$$d\rho(H_0) = H_0' \tag{2.2}$$

is called the H_2-*condition*. If either of these is satisfied, then there exists a unique holomorphic map $\tau\colon X \to \mathfrak{S}(V,\beta)$ such that $\tau(o) = J$, and the pair (ρ,τ) is equivariant in the sense that

$$\tau(g\cdot x) = \rho(g)\cdot\tau(x) \qquad \text{for all } g \in G(\mathbb{R})^0,\ x \in X.$$

Let Γ be a torsion-free arithmetic subgroup of $G(\mathbb{Q})$, and L a lattice in V such that $\rho(\Gamma)L = L$. The natural map

$$\mathcal{A} := (\Gamma \ltimes_\rho L)\backslash(X \times V_{\mathbb{R}}) \longrightarrow \mathcal{V} := \Gamma\backslash X$$

is a morphism of smooth quasiprojective algebraic varieties (Borel [9, Theorem 3.10, p. 559] and Deligne [10, p. 74]), so that $\mathcal{A}$ is a fiber variety over $\mathcal{V}$ called a *Kuga fiber variety.* The fiber $\mathcal{A}_P$ over any point $P \in \mathcal{V}$ is an abelian variety isomorphic to the torus $V_{\mathbb{R}}/L$ with the complex structure $\tau(x)$, where x is a point in X lying over P.

2.2. **Two algebraic groups.** We recall two algebraic groups associated to an abelian variety A over $\mathbb{C}$. Let $V = H_1(A,\mathbb{Q})$, and let β be an alternating Riemann form for A. The *Hodge group* $G(A)$, and *Lefschetz group* $L(A)$ are reductive $\mathbb{Q}$-subgroups of $GL(V)$. The Hodge group (or *special Mumford-Tate group*) is characterized by the property that for any positive integer k, the invariants of the action of $G(A)$ on $H^\bullet(A^k,\mathbb{Q})$ form the ring $\mathcal{H}(A^k)$ of Hodge classes (Mumford [14]). The Lefschetz group is defined to be the centralizer of $\mathrm{End}(A)$ in $Sp(V,\beta)$; it is characterized by the property that for any positive integer k, the subring of $\mathcal{H}(A^k)$ generated by the classes of divisors equals $H^\bullet(A^k,\mathbb{C})^{L(A)_\mathbb{C}} \cap H^\bullet(A^k,\mathbb{Q})$ (Milne [13, Theorem 3.2, p. 656] and Murty [16, §3.6.2, p. 93]). Clearly,

$$G(A) \subset L(A) \subset Sp(V,\beta).$$

The inclusion $G'(A) \hookrightarrow Sp(V,\beta)$ satisfies the H_1-condition (2.1) with respect to suitable H-elements. Taking $L = H_1(A,\mathbb{Z})$, and Γ to be any torsion-free arithmetic subgroup of $G'(A)$ such that $\gamma(x) \in L$ for all $\gamma \in \Gamma$, $x \in L$, we obtain a Kuga fiber variety having A as a fiber; this is called the *Hodge family* determined by A (see Mumford [14]).

The inclusion $L'(A)^0 \hookrightarrow Sp(V,\beta)$ satisfies the H_1-condition, and hence may be used to define a Kuga fiber variety having A as a fiber. These families of abelian varieties are generalizations of the PEL-families constructed by Shimura in [21].

Definition 1. An abelian variety A is of PEL-*type* if $G'(A) = L'(A)^0$.

Lemma 2. *Let A be an abelian variety of* PEL-*type. Suppose*

$$A = A_1^{n_1} \times A_2^{n_2} \times \cdots \times A_t^{n_t},$$

where $A_1, \dots, A_t$ are pairwise nonisogenous simple abelian varieties. Then, each A_i is of PEL*-type, and,*

$$G'(A) = G'(A_1) \times \cdots \times G'(A_t).$$

Reference to proof. See the first part of the proof of [2, Theorem 5.1, p. 348]. □

Proposition 3. *For any abelian variety A, the center of $G(A)$ is contained in the center of $L(A)$.*

Proof. Let $V = H_1(A, \mathbb{Q})$, let β be an alternating Riemann form for A, and, let Z be the center of $G(A)$. Since $D(A)$ is the centralizer of $G(A)$ in $\mathrm{End}(V)$, it follows that $Z \subset D(A)$. Since $L(A)$ is (by definition) the centralizer of $D(A)$ in $Sp(V, \beta)$, we see that Z is in the center of $L(A)$. □

3. Semidomination

We say that a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ of weight n is *fully twisted* if it is effective, and, $V^{n,0} \neq 0$. We say that a smooth, projective algebraic variety A over $\mathbb{C}$ is *dominated* by a set $\mathcal{X}$ of smooth, projective algebraic varieties over $\mathbb{C}$ if, for any irreducible Hodge structure V in the cohomology of A, there exists a fully twisted Hodge structure V' in the cohomology of some $X \in \mathcal{X}$ such that V' is isomorphic to a Tate twist of V.

Definition 4. We say that an abelian variety A is *semidominated* by a set $\mathcal{X}$ of abelian varieties if, given any nontrivial irreducible representation ρ of $G'(A)_{\mathbb{C}}$ such that ρ occurs in $H^n(A, \mathbb{C})$ for some n, there exist $A_\rho \in \mathcal{X}$, a positive integer c_ρ, and, $V_\rho \subset H^{c_\rho}(A_\rho, \mathbb{C})$, such that

- V_ρ is a $G(A_\rho)_{\mathbb{C}}$-submodule of $H^{c_\rho}(A_\rho, \mathbb{C})$,
- the action of $G'(A \times A_\rho)_{\mathbb{C}}$ on V_ρ is equivalent to $\rho \circ p_1$, where

  $$G'(A) \times G'(A_\rho) \supset G'(A \times A_\rho) \xrightarrow{p_1} G'(A)$$

  is the projection to the first factor,
- for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, the conjugate $(V_\rho)^\sigma$ contains a nonzero $(c_\rho, 0)$-form.

Definition 5. We say that an abelian variety A is *stably semidominated* by a set $\mathcal{X}$ of abelian varieties if every power of A is semidominated by $\mathcal{X}$.

Remark 6. We will see in Theorem 8 below that if A is semidominated by $\mathcal{X}$, then, A is dominated by abelian varieties. However, an abelian variety may be dominated by abelian varieties without being semidominated by any set of abelian varieties. For example, let A be a simple 4-dimensional abelian variety of type III. If A is of PEL-type, then, A is dominated by the set of powers of itself (see [5, Theorem 4.1 and Corollary 4.3]), but, it is not semidominated by any set of abelian varieties.

Lemma 7. *If A is semidominated by $\mathcal{X}$ and B is semidominated by $\mathcal{Y}$, and if $G'(A \times B) = G'(A) \times G'(B)$, then, $A \times B$ is semidominated by*

$$\mathcal{X} \cdot \mathcal{Y} = \{X \times Y \mid X \in \mathcal{X},\ Y \in \mathcal{Y}\}.$$

Proof. Any irreducible representation of $G'(A \times B)_{\mathbb{C}}$ is of the form $\rho \otimes \tau$, where ρ is an irreducible representation of $G'(A)_{\mathbb{C}}$ and τ is an irreducible representation of $G'(B)_{\mathbb{C}}$. Let

$$V_{\rho\otimes\tau} = \begin{cases} V_\rho \otimes V_\tau & \text{if both } \rho \text{ and } \tau \text{ are nontrivial;} \\ V_\rho & \text{if } \rho \text{ is nontrivial but } \tau \text{ is trivial;} \\ V_\tau & \text{if } \tau \text{ is nontrivial but } \rho \text{ is trivial.} \end{cases}$$

□

Theorem 8. *Let A be an abelian variety semidominated by $\mathcal{X}$. Then, A is dominated by the set of abelian varieties of the form $B \times C$, where $B \in \mathcal{X}$, and C is of CM-type.*

Proof. Let V be any irreducible Hodge structure in the cohomology of A. If $G'(A)$ acts trivially on V, then, V is of CM-type, so by [7, Theorem 3, p. 159] there exists an abelian variety C of CM-type, and a fully twisted Hodge structure V' in the cohomology of C, such that V' is isomorphic to a Tate twist of V. Otherwise, let V_0 be any irreducible $G(A)_{\mathbb{C}}$ submodule of $V_{\mathbb{C}}$; it is necessarily irreducible as a $G'(A)_{\mathbb{C}}$-module. Since A is semidominated by $\mathcal{X}$, there exists $B \in \mathcal{X}$, and an irreducible $G(B)_{\mathbb{C}}$-submodule V_ρ of $H^{c_\rho}(B, \mathbb{C})$ satisfying the conditions of Definition 4.

Let T be the torus $G(A \times B)/G'(A \times B)$. Since every representation of $G(A \times B)$ occurs in the tensor algebra of $H^1(A \times B, \mathbb{Q})$ [10, Proposition 3.1(a), p. 40], there exists a Hodge structure W in the cohomology of some power of $A \times B$ such that the action of $G(A \times B)$ on W is equivalent to the representation $G(A \times B) \to T$. Then $G(W) = T$, so, W is of CM-type.

V_0 is equivalent to V_ρ as a $G'(A)_{\mathbb{C}}$-module. Therefore V_0 is equivalent to $V_\rho \otimes \chi$ as a $G(A)_{\mathbb{C}}$-module, where χ is a character of $T_{\mathbb{C}}$. The character χ occurs in the tensor algebra of W. Let W_χ be its representation space, and let Z be an irreducible Hodge structure in the tensor algebra of W such that $Z_{\mathbb{C}}$ contains W_χ.

By the main theorem of [7], there exists an abelian variety C of CM-type and an irreducible Hodge structure $Z' \subset H^c(C, \mathbb{Q})$ such that Z' is isomorphic to a Tate twist $Z(w)$ of Z, and, Z' is fully twisted. Let $\varphi\colon Z(w) \to Z'$ be an isomorphism of Hodge structures, and let $Z'_\chi = \varphi(W_\chi)$. Then there exists $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that $(Z'_\chi)^\sigma$ contains a nonzero $(c, 0)$-form. Let

$$U' = V_\rho \otimes Z'_\chi \subset H^{c_\rho + c}(B \times C, \mathbb{C}).$$

Let $\widetilde{V}'$ be the smallest Hodge substructure of $H^{c_\rho+c}(B \times C, \mathbb{Q})$ such that $\widetilde{V}'_{\mathbb{C}}$ contains U'. Then, $\widetilde{V}'$ is fully twisted because U'^σ contains a nonzero $(c_\rho + c, 0)$-form. Recall that V is an irreducible $G(A)$-module, and note that

$\widetilde{V}'$ is a primary $G(B \times C)$-module. Let $G = G(A \times B \times C)$. We have a $G_{\mathbb{C}}$-isomorphism between V_0 and U', so $\hom_{G_{\mathbb{C}}}(V_{\mathbb{C}}, \widetilde{V}'_{\mathbb{C}})$ is nontrivial. Since $G(\mathbb{Q})$ is Zariski-dense in $G(\mathbb{C})$ (Rosenlicht [17, Corollary, p. 44]), we have

$$\hom_{G(\mathbb{C})}(V_{\mathbb{C}}, \widetilde{V}'_{\mathbb{C}}) = \hom_{G(\mathbb{Q})}(V_{\mathbb{C}}, \widetilde{V}'_{\mathbb{C}}) = \hom_G(V, \widetilde{V}') \otimes \mathbb{C}.$$

Therefore $\hom_G(V, \widetilde{V}')$ is nontrivial, and V is isomorphic to a Tate twist of an irreducible Hodge structure V' contained in $\widetilde{V}'$. □

4. Abelian varieties associated to $SU(p, 1)$

Let A be a simple abelian variety. We denote by $\alpha \mapsto \alpha'$ the Rosati involution induced on the division algebra $D = D(A)$ by an alternating Riemann form β for A. We assume that A is of type IV, i.e., the center K of D is a CM-field, and, we denote by F the maximal totally real subfield of K. We consider $V = H_1(A, \mathbb{Q})$ as a left D-module. There exists a unique F-bilinear form $T \colon V \times V \to D$ such that

$$\begin{aligned} \beta(x, y) &= \operatorname{Tr}_{D/\mathbb{Q}} T(x, y), \\ T(ax, by) &= aT(x, y)b', \\ T(y, x) &= -T(x, y)', \end{aligned}$$

for all x, $y \in V$, and all a, $b \in D$ (Shimura [22, Lemma 1.2, p. 162]).

The unitary group $U(V, T) = \operatorname{Aut}_D(V, T)$ is then a reductive algebraic group over F, and, the Lefschetz group of A is given by

$$L(A) = \operatorname{Res}_{F/\mathbb{Q}} U(V, T).$$

Let S be the set of embeddings of F into $\mathbb{R}$, and $m = \dim_K V$. Then we can write $L(A)_{\mathbb{R}} = \prod_{\alpha \in S} L_\alpha$ and $V_{\mathbb{R}} = \bigoplus_{\alpha \in S} V_\alpha$ where $L_\alpha \cong U(p_\alpha, q_\alpha)$ acts trivially on $V_{\alpha'}$ unless $\alpha = \alpha'$, and, $L_{\alpha,\mathbb{C}} \cong GL_m(\mathbb{C})$ acts on $V_{\alpha,\mathbb{C}}$ as the direct sum of the standard representation and its contragredient (Murty [15]).

Let $L' = L'(A)$ be the derived group of $L(A)$. Then, $L'(\mathbb{R}) = \prod_{\alpha \in S} L'_\alpha$, where $L'_\alpha \cong SU(p_\alpha, q_\alpha)$.

Theorem 9 (Satake [18, 19])**.** *With the above notations, assume that for each $\alpha \in S$, we have $q_\alpha = 1$. Then, for each $k = 1, \ldots, m-1$, there exists a homomorphism of $\mathbb{Q}$-algebraic groups*

$$\rho_k \colon L' \to Sp(V_k, \beta_k), \tag{4.1}$$

where β_k is a nondegenerate alternating form on a finite-dimensional $\mathbb{Q}$-vector space V_k, ρ_k satisfies the H_1-condition (2.1)*, and, ρ_k is equivalent over $\mathbb{R}$ to the direct sum of $\bigoplus_{\alpha \in S} \bigwedge^k \circ P_\alpha$ and its contragredient, where $P_\alpha \colon L'(\mathbb{R}) \to L'_\alpha$ is the projection.*

Remark 10. We make the following remarks about Theorem 9 and its proof. Let $\mathcal{A}_k \to \mathcal{V}$ be a Kuga fiber variety defined by the symplectic representation (4.1).

(1) $\mathcal{A}_1 \to \mathcal{V}$ is a PEL-family determined by A.
(2) Since the contragredient of $\bigwedge^k$ is $\bigwedge^{m-k}$, we see that ρ_k and ρ_{m-k} are equivalent.
(3) Since $L'(\mathbb{R})$ has no compact factors, it follows from [1, Remark 3.5, p. 213] that each $\mathcal{A}_k \to \mathcal{V}$ is a Hodge family.
(4) ρ_k factors as

$$L' \xrightarrow{\tilde{\rho}_k} L_1 \xrightarrow{\tilde{\varphi}_k} Sp(V_k, \beta_k),$$

where $L_1 = \operatorname{Res}_{F/\mathbb{Q}} \widetilde{L}_1$ for a semisimple algebraic group $\widetilde{L}_1$ over F. We have $L_1(\mathbb{R}) = \prod_{\alpha \in S} L_{1,\alpha}$, where $L_{1,\alpha} \cong SU(p', q')$. Here, $p' = \binom{p}{k}$ and $q' = \binom{p}{k-1}$.
(5) $\tilde{\rho}_{k,\mathbb{R}} \colon L'(\mathbb{R}) \to L_1(\mathbb{R})$ can be described as follows: for $g = (g_\alpha) \in L'(\mathbb{R}) = \prod_{\alpha \in S} L'_\alpha$, we have

$$\tilde{\rho}_{k,\mathbb{R}}(g) = \tilde{\rho}_{k,\alpha}(g_\alpha) \in L_1(\mathbb{R}) = \prod_{\alpha \in S} L_{1,\alpha},$$

where $\tilde{\rho}_{k,\alpha} \colon L'_\alpha = SU(p,1) \to L_{1,\alpha} = SU(p',q')$ is equivalent to the representation on the k-th exterior power.
(6) We can write $V_{k,\mathbb{R}} = \bigoplus_{\alpha \in S} V_{k,\alpha}$, such that β_k is nondegenerate on each $V_{k,\alpha}$, $L_{1,\alpha}$ acts trivially on $V_{k,\alpha'}$ unless $\alpha = \alpha'$, and, the action of $L_{1,\alpha}(\mathbb{C})$ on $V_{k,\alpha,\mathbb{C}}$ is equivalent to the direct sum of the standard representation and its contragredient.

Theorem 11. *Let A be an abelian variety of* PEL*-type such that each simple factor of A is of type IV, and, each simple factor of $G'(A)(\mathbb{R})$ is isomorphic to $SU(p,1)$ for some p. Then A is dominated by abelian varieties.*

Proof. Thanks to Theorem 8, it suffices to show that A is stably semidominated by some set of abelian varieties. Since A is of PEL-type, it follows from Lemma 2 that

$$G'(A) = G'(B_1) \times \cdots \times G'(B_t),$$

where $B_1, \ldots, B_t$ are the distinct simple factors of A, up to isogeny. So, by Lemma 7, we may assume that A is a simple abelian variety.

Let $\mathcal{A} \to \mathcal{V}$ be the Hodge family of A, and let $P \in \mathcal{V}$ be such that $A = \mathcal{A}_P$. For each $k = 1, \ldots, p$, there is a Kuga fiber variety $\mathcal{A}_k \to \mathcal{V}$ defined by the symplectic representation ρ_k (4.1). Let $A_k = (\mathcal{A}_k)_P$. We will show that A is stably semidominated by the set

$$\mathcal{X} = \left\{ A_1^{n_1} \times \cdots \times A_p^{n_p} \mid n_i \geq 0 \right\}.$$

We claim that $G'(A \times X) = G'(A)$ for any $X \in \mathcal{X}$. More precisely, we mean that the projection $G(A) \times G(X) \to G(A)$, restricted to the subgroup $G'(A \times X)$ of $G(A) \times G(X)$ induces an isomorphism of $G'(A \times X)$ with $G'(A)$. To see this, let $X = A_1^{n_1} \times \cdots \times A_p^{n_p}$. Then,

$$A \times X = A_1^{n_1+1} \times A_2^{n_2} \times \cdots \times A_p^{n_p},$$

since $A = A_1$. Thus $A \times X$ is the fiber over P of the Kuga fiber variety

$$\mathcal{B} = \mathcal{A}_1^{n_1+1} \times_{\mathcal{V}} \mathcal{A}_2^{n_2} \times_{\mathcal{V}} \cdots \times_{\mathcal{V}} \mathcal{A}_p^{n_p} \to \mathcal{V}$$

defined by the symplectic representation

$$\rho = (n_1 + 1)\rho_1 \oplus n_2\rho_2 \oplus \cdots \oplus n_p\rho_p$$

of L'. This is a Hodge family (see Remark 10.3). Hence $G'(\mathcal{B}_Q) \subset L'$ for all $Q \in \mathcal{V}$. But $L' = G'(A)$ is a quotient of $G'(A \times X) = G'(\mathcal{B}_P)$. Therefore $G'(A \times X) = L' = G'(A)$.

We note that if $p = 0$, then a result of Shimura [21, Proposition 14, p. 176] implies that A is of CM-type, so $G'(A)$ is trivial and there is nothing to prove. If $p = 1$, then $G(A)$ is semisimple, and the theorem is a special case of [2, Theorem 5.1, p. 348]. We therefore assume $p > 1$.

We have $L_{\alpha,\mathbb{C}} \cong GL_m(\mathbb{C})$, where $m = p + 1$. As explained in [2, p. 351], $V_{\alpha,\mathbb{C}} = Y_\alpha \oplus \overline{Y}_\alpha$, where Y_α and its complex conjugate $\overline{Y}_\alpha$ are $L_{\alpha,\mathbb{C}}$-modules (and therefore $G(A)_{\mathbb{C}}$-modules); $GL_m(\mathbb{C})$ acts on Y_α as the standard representation, and on $\overline{Y}_\alpha$ as the contragredient. Y_α is the direct sum of a p-dimensional space of $(1, 0)$-forms and a 1-dimensional space of $(0, 1)$-forms. $\overline{Y}_\alpha$ is the direct sum of a 1-dimensional space of $(1, 0)$-forms and a p-dimensional space of $(0, 1)$-forms. Choose a basis $\{u_1, \ldots, u_m\}$ of Y_α such that $u_1, \ldots, u_p$ are $(1, 0)$-forms and u_{p+1} is a $(0, 1)$-form. Then $\{\overline{u}_1, \ldots, \overline{u}_m\}$ is a basis of $\overline{Y}_\alpha$. Observe that the set $\bigcup_{\alpha\in S} \{Y_\alpha, \overline{Y}_\alpha\}$ is invariant under the action of $\mathrm{Aut}(\mathbb{C})$, so every Galois conjugate of Y_α contains a nonzero $(1, 0)$-form.

Let $\mu_1, \ldots, \mu_{m-1}$ be the fundamental weights of $SL_m(\mathbb{C})$, i.e., μ_k is the highest weight of the representation $\bigwedge^k(\mathrm{St})$, where (St) denotes the standard representation of $SL_m(\mathbb{C})$ on $\mathbb{C}^m$.

For $1 \le k \le p$, let $V_k := H^1(A_k, \mathbb{Q})$. We can write

$$V_{k,\mathbb{R}} = \bigoplus_{\alpha\in S} V_{k,\alpha}, \tag{4.2}$$

where, L'_α acts trivially on $V_{k,\alpha'}$ unless $\alpha = \alpha'$. Then $V_{k,\alpha,\mathbb{C}} = Y_{k,\alpha} \oplus \overline{Y}_{k,\alpha}$, where $Y_{k,\alpha} = \bigwedge^k Y_\alpha$ and its complex conjugate $\overline{Y}_{k,\alpha} = \bigwedge^k \overline{Y}_\alpha$ are $L_{\alpha,\mathbb{C}}$-modules, $GL_m(\mathbb{C})$ acts on $Y_{k,\alpha}$ as $\bigwedge^k(\mathrm{St})$, and on $\overline{Y}_{k,\alpha}$ as the contragredient. We have $G'(A_k) = \rho_k(G'(A))$, so $Y_{k,\alpha}$ and $\overline{Y}_{k,\alpha}$ are $G'(A_k)_{\mathbb{C}}$-modules.

$Y_{k,\alpha}$ is the direct sum of a $\binom{p}{k}$-dimensional space of $(1, 0)$-forms and a $\binom{p}{k-1}$-dimensional space of $(0, 1)$-forms. In particular, the highest weight vector in $Y_{k,\alpha}$ is

$$w_k := u_1 \wedge \cdots \wedge u_k$$

which is a $(1, 0)$-form, while

$$w'_k := u_{m-k+1} \wedge \cdots \wedge u_m$$

is a $(0, 1)$-form. $\overline{Y}_{k,\alpha}$ is the direct sum of a $\binom{p}{k-1}$-dimensional space of $(1, 0)$-forms and a $\binom{p}{k}$-dimensional space of $(0, 1)$-forms. Observe that the set

$\bigcup_{\alpha\in S}\left\{Y_{k,\alpha},\overline{Y}_{k,\alpha}\right\}$ is invariant under the action of $\mathrm{Aut}(\mathbb{C})$, so every Galois conjugate of $Y_{k,\alpha}$ contains a nonzero $(1,0)$-form.

Let j be a positive integer. Then $S^jY_{k,\alpha}$, the symmetric tensors on $Y_{k,\alpha}$, is a representation of $SL_m(\mathbb{C})$ with highest weight $j\mu_k$, and highest weight vector $(w_k)^j$. Let $V^j_{k,\alpha}\subset H^j(A^j_k,\mathbb{C})$ be the $SL_m(\mathbb{C})$-module generated by $(w_k)^j$. The highest weight vector in $V^j_{k,\alpha}$ is $(w_k)^j$ which is a $(j,0)$-form. Thus $V^j_{k,\alpha}$ is an irreducible representation of $SL_m(\mathbb{C})$ with highest weight $j\mu_k$, which contains both the $(j,0)$-form $(w_k)^j$ and the $(0,j)$-form $(w'_k)^j$. Observe that the set

$$\left\{\,V^j_{k,\alpha}\mid \alpha\in S,\ 1\le k\le p,\ j>0\,\right\}\cup\left\{\,\overline{V}^j_{k,\alpha}\mid \alpha\in S,\ 1\le k\le p,\ j>0\,\right\}$$

is invariant under the action of $\mathrm{Aut}(\mathbb{C})$, so every Galois conjugate of $V^j_{k,\alpha}$ contains a nonzero $(j,0)$-form.

Any irreducible representation π of $SL_m(\mathbb{C})$ has highest weight

$$\mu=a_1\mu_1+\cdots+a_p\mu_p$$

where the a_j are nonnegative integers. Let $a=\sum_{k=1}^p a_k$. Then the representation

$$\bigotimes_{k=1}^{p}V^{a_k}_{k,\alpha}\subset H^a(A_1^{a_1}\times\cdots\times A_p^{a_p},\mathbb{C})$$

has highest weight μ. The vector $v_\mu:=\bigotimes_{k=1}^p(w_k)^{a_k}$ generates an irreducible submodule V^α_μ which has highest weight μ. Note that V^α_μ contains both the $(a,0)$-form v_μ and the $(0,a)$-form $v'_\mu:=\bigotimes_{k=1}^p(w'_k)^{a_k}$. Observe that the set

$$\left\{\,V^\alpha_\mu\;\middle|\;\alpha\in S,\ \mu=\sum_{i=1}^p a_i\mu_i, a_i\ge 0\,\right\}$$

is invariant under the action of $\mathrm{Aut}(\mathbb{C})$, so every Galois conjugate of V^α_μ contains a nonzero $(a,0)$-form.

Any irreducible representation ρ of $G'_{\mathbb{C}}$ is of the form $\rho=\bigotimes_{\alpha\in S}\pi_\alpha$, where π_α is an irreducible representation of $G'_{\alpha,\mathbb{C}}\cong SL_m(\mathbb{C})$. Let

$$V_\rho=\bigotimes_{\alpha\in S}V_{\pi_\alpha}.$$

Then V_ρ is an irreducible $G'_{\mathbb{C}}$-submodule of $H^c(A_\rho,\mathbb{C})$, for some $A_\rho\in\mathcal{X}$ and some positive integer c, on which $G'_{\mathbb{C}}$ acts as ρ, and which contains both nonzero $(c,0)$-forms and nonzero $(0,c)$-forms. Since every Galois conjugate of each V_{π_α} contains a nonzero holomorphic form, it follows that every Galois conjugate of V_ρ contains a nonzero $(c,0)$-form.

To complete the proof that A is stably semidominated by $\mathcal{X}$, we need to show that the $G'(A_\rho)_{\mathbb{C}}$-module V_ρ is actually a $G(A_\rho)_{\mathbb{C}}$-module. By Proposition 3, it is sufficient to show that V_ρ is a $Z\big(L(A_\rho)\big)_{\mathbb{C}}$-module. Write

$$A_\rho=A_1^{n_1}\times\cdots\times A_p^{n_p}.$$

Assume, without loss of generality, that each $n_k > 0$. Since A_k and A_{m-k} are isomorphic (Remark 10.2), we have

$$L(A_\rho) = \prod_{2k \le m} L(A_k).$$

If $k = m/2$, then, $G(A_k)$ and $L(A_k)$ are semisimple, so

$$Z\big(L(A_\rho)\big) = \prod_{1 \le k < \frac{m}{2}} Z\big(L(A_k)\big).$$

Now, for $k \neq m/2$,

$$Z\big(L(A_k)\big)_{\mathbb{C}} = \prod_{\alpha \in S} Z_{k,\alpha}$$

where each $Z_{k,\alpha} \cong \mathbb{C}^\times$, and in the decomposition (4.2), $Z_{k,\alpha}$ acts trivially on $V_{k,\alpha'}$ unless $\alpha = \alpha'$. A scalar $z_\alpha \in Z_{k.\alpha}$ acts on $Y_{k,\alpha}$ as multiplication by z_α, and on $\overline{Y}_{k,\alpha}$ as multiplication by $\overline{z}_\alpha$; thus $Y_{k,\alpha}$ and $\overline{Y}_{k,\alpha}$ are both $Z_{k,\alpha}$-modules. The scalar z_α acts as multiplication by z_α^j on $V^j_{k,\alpha}$; so $V^j_{k,\alpha}$ is a $Z_{k,\alpha}$-module. Now let

$$z = (z_{k,\alpha}) \in Z\big(L(A_\rho)\big)_{\mathbb{C}} = \prod_{1 \le k < \frac{m}{2}} \prod_{\alpha \in S} Z_{k,\alpha}.$$

Then z acts on $\bigotimes_{k=1}^p V^{a_k}_{k,\alpha}$ as multiplication by $\prod_{2k<m} z_{k,\alpha}^{a_k + a_{m-k}}$. Hence any subspace of it (in particular, V^α_μ) is a $Z\big(L(A_\rho)\big)_{\mathbb{C}}$-module. It follows that $V_\rho = \bigotimes_{\alpha \in S} V_{\pi_\alpha}$ is a $Z\big(L(A_\rho)\big)_{\mathbb{C}}$-module. □

Remark 12. The proof of the above theorem actually shows that each member of $\mathcal{X}$ is stably semidominated by $\mathcal{X}$.

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